Pseudo-reductive groups

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The course will consist of 4 lectures of 1.5 hours each on examples, structure, and applications of pseudo-reductive groups.

After some preliminary motivation for the study of pseudo-reductivity, we will see that the internal structure theory of such groups has many similarities to the reductive case (e.g., root groups, relative root systems, open cell, and Bruhat decomposition) but often requires rather different proofs (e.g., root groups must be developed in an entirely different manner, and SL_2 does not play the same central role as in the reductive case but it remains important). Such groups can also exhibit features that deviate from the reductive case (e.g., nontrivial central etale p-torsion in characteristic p, and loss of pseudo-reductivity under smooth quotients).

The key to the usefulness of the theory (and the ingredient missed in the early work of Borel and Tits on this topic) is the "standard construction", and the main aim of the lectures is to explain the proof of exhaustiveness of the standard construction over fields not of characteristics 2 or 3, as well as the additional possibilities that exist in these small characteristics. This provides a very satisfactory classification "modulo the commutative case", and we will illustrate some ways in which the ubiquity of standardness can be applied (and how one handles non-standard examples in small characteristic).

Prerequisites

Familiarity with the structure theory of reductive groups over algebraically closed fields, basic scheme theory (e.g., flatness, smooth and étale morphisms, the functorial perspective), and special features of affine algebraic groups in positive characteristic (e.g., Frobenius kernel and quotients/exactness via the fppf topology). Some familiarity descent theory will be useful (especially Galois descent).

References

Bosch, S., Lütkebohmert, W., Raynaud, M. — *Néron models*, Ergebnisse in Mathematik und ihre Grenzgebite, Band 21, Springer Verlag, 1990.
(*Chapter 2 reviews smooth and etale morphisms, and sections 6.1-6.2 develop the basics of descent theory and its relation with Galois descent. We mainly need the affine case, which is simpler.*)

Conrad, B., Gabber, O., Prasad, G. — *Pseudo-reductive groups*, Cambridge University Press, 2010.

Rémy, B. — Groupes algébriques pseudo-réductifs et applications, Sém. Bourbaki 1021, 2009-10.

(*This provides an excellent overview of the general theory, some examples and specific applications.*)